Chapter 1
Empirical Input–Output Representations and Transmutations of Actual Economies

Abstract This chapter maps the structure of the empirical input–output representations of actual economies, i.e. the Supply and Use Tables and the Symmetric Input–Output Tables, and critically evaluates the methods that have been proposed to convert the former into the latter. It is argued that (i) all conversion methods rest on the groundless assumption that single production, and not joint production, characterizes the economic structure of the real world; and (ii) a consistent approach is the straightforward treatment of the Supply and Use Tables on the basis of general joint production models inspired by the von Neumann and Sraffa contributions.

Keywords Input–output analysis • Joint production • Supply and use tables • Symmetric input–output tables • von Neumann–Sraffa-based analysis

1.1 Introduction

The representation of actual economies in terms of input–output relationships, where the inter-dependency amongst the different production activities plays a central role, dates back to the *Tableau Économique* of François Quesnay (1972). In this table, Quesnay distinguished two productive sectors (primary production and manufacturing) and three social classes, i.e. the "productive class" (*classe productive*), which is involved in primary production; the "sterile class" (*classe stérile*), involved in manufacturing; and the class of proprietors of land and natural resources (*classe propriétaire*). The modern representation of actual economies and the development of input–output analysis as a distinct area of economics was introduced by Wassily Leontief (1936), who constructed the *tableau économique* or, in modern terms, the input–output table for the economy of the United States of America.[1]

During the last decades, there has been a significant development, both theoretically and empirically, of input–output analysis, and, today, input–output tables constitute part of the national accounting systems for most countries. The most known form of input–output tables, and the most widely used in empirical studies, are the so-called Symmetric Input–Output Tables (SIOTs). SIOTs represent the intersectoral relationships of an economy in which the number of products equals the number of production activities and each product is produced by only one production activity; therefore, SIOTs rule out, by construction, joint production. On the other hand, a less often used form of input–output tables, for empirical applications, are the Supply and Use Tables (SUTs), which constitute a pair of tables: one describes the production of goods and services by the different industries (Supply Table), and the other describes the use of goods and services by the different industries (Use Table). Contrary to the structure of the SIOTs, in

[1] For a review of the contributions to the foundation of input–output analysis from the Physiocrats to Piero Sraffa, see Kurz et al. (1998). For the *Tableau Économique*, also see Marx ([1878] 1977).

the SUTs, there are industries that produce more than one product and products that are produced by more than one industry; therefore, these tables do not rule out joint production. Furthermore, SUTs constitute the core of the modern system of national accounting and also the basis for the derivation of SIOTs under specific assumptions.

This chapter (i) presents the basic characteristics of both the SUTs and SIOTs; (ii) critically reviews the methods that have been proposed to convert SUTs into SIOTs; and (iii) exposes the essential ideas of the von Neumann–Sraffa-based approach on joint production as a preferable approach to treat SUTs. Section 1.2 presents the basic structure of the SUTs. Section 1.3 presents the basic structure of the SIOTs. Section 1.4 critically reviews and evaluates the methods that have been proposed to convert SUTs into SIOTs. Section 1.5 presents the essential ideas of the von Neumann–Sraffa- based approach on joint production as a way to treat SUTs. Finally, Sect. 1.6 concludes.

1.2 The Supply and Use Tables

In 1968 System of National Accounts, United Nations introduced the supply and use framework in the compilation of national accounts (United Nations 1968).[2] This framework forms the basis for the most detailed description of a national economy, providing information about the supply and demand side of the economic system as well as its relations with other national economies. The core of the supply and use framework consists of a pair of tables, known as the Supply and Use Tables (SUTs hereafter). The SUTs describe the flows of goods and services produced by the different industries of a national economy, the flows of goods and services with the rest of the world, the structure of the cost of production of each industry, the income generated in the production processes, and the final uses in the economy. The SUTs provide detailed information not only regarding the inter-dependencies amongst the various industries of the national economy but also on basic macroeconomic aggregates, such as gross domestic product, value added, total and intermediate consumption, capital formation, exports, and imports.

The supply table describes the production of goods and services by the different industries, distinguishing domestic supply from imports per product. The part of the supply table that describes domestic production of the different industries is called the "make matrix" of the economy. A simplistic supply table, describing the production of two commodities by two industries is shown in Table 1.1. Thus, the make matrix, $\mathbf{M} \equiv [M_{ij}]$ of this particular economy is of dimensions 2×2.[3]

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[2] What follows draws on Soklis (2005; 2012, Chap. 4).

[3] In general, the SUTs need not be "square", i.e. the number of goods and services produced need not be equal to the producing industries (see, e.g. Eurostat 2008, p. 325; United Nations 1999, p.

Table 1.1 Simplistic Supply Table

Products \ Industries	Industry 1	Industry 2	Total	Imports	Total Supply
Product 1	M_{11}	M_{12}	$\sum_{j=1}^{2} M_{1j}$	IM_1	TS_1
Product 2	M_{21}	M_{22}	$\sum_{j=1}^{2} M_{2j}$	IM_2	TS_2
Total	$\sum_{i=1}^{2} M_{i1}$	$\sum_{i=1}^{2} M_{i2}$	$\sum_{i,j=1}^{2} M_{ij}$	$\sum_{i=1}^{2} IM_i$	TS

Each row of the make matrix gives the quantities (in money terms) of each product produced by the different industries, while each column gives the quantities of the different products produced by each industry. Thus, M_{ij} denotes the quantity of product i produced by industry j. The on-diagonal elements of the make matrix describe the so-called "primary (or characteristic) product" of each industry and the off-diagonal elements describe the so-called "secondary products", where the "primary product" of an industry is defined as the output of that industry that comprises the primary source of revenues. The column of imports gives the quantities of total imports of each commodity, while the last column of the supply table gives the total supply of the economy (domestic production plus imports) of each commodity. The last row of the supply table gives the total output of each industry, the total imports and the total supply in the economy. Thus, IM_i denotes the imports of product i; TS_i the total supply of product i in the economy; and TS the total supply in the economy.

The use table describes the use of goods and services by the different industries, the income generation per production activity, and the final uses of the production per category of final demand. The part of the use table that describes intermediate consumption by product and by industry is called the "use matrix" of the economy. A simplistic use table, describing the uses of two commodities by two industries is shown in Table 1.2. Thus, the use matrix, $\mathbf{U} \equiv [U_{ij}]$ of this particular economy is of dimensions 2×2.

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86). Nevertheless, in what follows we assume, for simplicity's sake, that the make and use matrices are square.

Table 1.2 Simplistic Use Table

Products \ Industries	Industry 1	Industry 2	Total	Final Uses	Total Uses
Product 1	U_{11}	U_{12}	$\sum_{j=1}^{2} U_{1j}$	F_1	TU_1
Product 2	U_{21}	U_{22}	$\sum_{j=1}^{2} U_{2j}$	F_2	TU_2
Total Intermediate Consumption	$\sum_{i=1}^{2} U_{i1}$	$\sum_{i=1}^{2} U_{i2}$	$\sum_{i,j=1}^{2} U_{ij}$	$\sum_{i=1}^{2} F_i$	TU
Value Added	V_1	V_2	$\sum_{j=1}^{2} V_j$		
Total Output	X_1	X_2	$\sum_{j=1}^{2} X_j$		

Each row of the use matrix gives the quantities (in money terms) of each product used by the different industries, while each column gives the quantities of the different products used by each industry. Thus, U_{ij} denotes the quantity of product i used by industry j. The column of final uses gives the uses of products for final consumption, gross capital formation and exports, while the row of value added gives the components of value added per industry, i.e. compensation of employees, other net taxes on production, consumption of fixed capital and net operating surplus. The last column of the use table gives the total uses (intermediate consumption plus final uses) by product, while the last row of the table shows the total inputs (intermediate consumption plus value added) by industry and, therefore, is identified with the money value of the total output of each industry. Thus, F_i denotes the quantity of product i used for final demand; TU_i the total use of product i in the economy; V_j the value added in industry j; X_j the total output of industry j; and TU the total uses in the economy (all data are expressed in monetary units).

By construction of the SUTs, the following identities hold for the supply table

$$\sum_{j=1}^{2} M_{ij} + IM_i \equiv TS_i \tag{1.1}$$

$$\sum_{i,j=1}^{2} M_{ij} + \sum_{i=1}^{2} IM_i \equiv TS \tag{1.2}$$

From identities (1.1) and (1.2) it follows that it is also holds

$$\sum_{i=1}^{2} TS_i \equiv TS \tag{1.3}$$

By construction of the SUTs, the following identities hold for the use table

$$\sum_{j=1}^{2} U_{ij} + F_i \equiv TU_i \tag{1.4}$$

$$\sum_{i,j=1}^{2} U_{ij} + \sum_{i=1}^{2} F_i \equiv TU \tag{1.5}$$

$$\sum_{i=1}^{2} U_{ij} + V_j \equiv X_j \tag{1.6}$$

From identities (1.4) and (1.5) it follows that it also holds

$$\sum_{i=1}^{2} TU_i \equiv TU \tag{1.7}$$

The supply and the use tables are connected through the following identity

$$TS_i \equiv TU_i \tag{1.8}$$

By taking into account identities (1.1) and (1.4), identity (1.8) can be re-written as

$$\sum_{j=1}^{2} M_{ij} + IM_i \equiv \sum_{j=1}^{2} U_{ij} + F_i \tag{1.9}$$

Thus, taking into account identities (1.3) and (1.7), it also holds

$$TS \equiv TU$$

The supply and the use tables are also connected through the following identity

$$\sum_{i=1}^{2} M_{ij} \equiv X_j$$

or, by taking into account identity (1.6),

$$\sum_{i=1}^{2} M_{ij} \equiv \sum_{i=1}^{2} U_{ij} + V_j \tag{1.10}$$

The identities (1.9) and (1.10) can be re-written in vector–matrix terms as follows

$$\mathbf{Me} + \mathbf{Im} \equiv \mathbf{Ue} + \mathbf{f} \tag{1.11}$$

$$\mathbf{e}^{\mathrm{T}}\mathbf{M} \equiv \mathbf{e}^{\mathrm{T}}\mathbf{U} + \mathbf{v}^{\mathrm{T}} \tag{1.12}$$

where $\mathbf{M} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$ is the make matrix; $\mathbf{Im} = \begin{bmatrix} IM_1 \\ IM_2 \end{bmatrix}$ is the vector of imports; $\mathbf{U} = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}$ is the use matrix; $\mathbf{f} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$ is the vector of final demand; $\mathbf{v} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$ is the value added vector; $\mathbf{e} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ the column summation vector; and the superscript "T" denotes the transpose.

From the previous, it follows that SUTs constitute a general accounting framework that allows statistical authorities to enter basic economic data in the exact structure in which they are observed. It is interesting to note that despite the fact that SUTs give a very detailed "picture" of the actual economies, they are rarely used in input–output applications in the form presented above.[4] On the other hand, the usual basis for input–output applications are the Symmetric Input–Output Tables (SIOTs hereafter). The main difference between SUTs and SIOTs is that in the SUTs (SIOTs) there are (are no) industries that produce more than one commodity and (nor) commodities that are produced by more than one industry. Thus, the SUTs (SIOTs) allow for (exclude) joint production activities, which constitute a central characteristic of the actual economic world.[5] In the next section we present the main characteristics of the SIOTs.

1.3 The Symmetric Input–Output Tables

The SIOTs are constructed on the basis of assumptions on the relationships between inputs and outputs that have been recorded in the SUTs.[6] The alternative assumptions that have been proposed to convert SUTs into SIOTs will be analytically discussed in Sect. 1.4. Depending on the assumptions used to construct SIOTs, the derived input–output table can either describe the relationships amongst the products of a national economy ("product-by-product" SIOTs) or the relationships amongst the industries of the economy ("industry-by-industry" SIOTs). The choice between the construction of product-by-product tables or industry-by-industry tables depends on the analytical purposes that these tables are intended to be used. Industry-by-industry SIOTs are considered to be closer to statistical sources and actual market transactions, while product-by-product SIOTs are considered as more homogenous in terms of cost structures and production activities (see, e.g. Eurostat 2008, p. 24). The part of the SIOTs table that describes intermediate consumption either by product or by industry is called the "Transactions Matrix" or the "Matrix of Intermediate Inputs" of the economy. A simplistic product-by-product SIOT for an economy producing only

[4] Exceptions can be found in Mariolis and Soklis (2007; 2010; 2018); Mariolis et al. (2018); Soklis (2006; 2011; 2012; 2015). Also see Chaps. 5, 7 and 9 of this book.

[5] It has to be noted that some of the secondary products that appear in the SUTs may result from statistical classification. Therefore, these products do not correspond with the notion of joint production (see, e.g. United Nations 1999, p. 77). However, this fact by no means undermine the empirical importance of joint production (see, e.g. Baumgärtner et al. 2006; Faber et al. 1998; Kurz 2006; Steedman 1984).

[6] What follows draws on Soklis (2012, Chap. 3).

two products is shown in Table 1.3.[7] Thus, the transactions matrix, $\mathbf{Z} \equiv [Z_{ij}]$ of this particular economy is of dimensions 2×2.

Table 1.3 Simplistic Symmetric Input–Output Table

Products \ Products	Product 1	Product 2	Total	Final Uses	Total Uses
Product 1	Z_{11}	Z_{12}	$\sum_{j=1}^{2} Z_{1j}$	F_1	TU_1
Product 2	Z_{21}	Z_{22}	$\sum_{j=1}^{2} Z_{2j}$	F_2	TU_2
Total Intermediate Consumption	$\sum_{i=1}^{2} Z_{i1}$	$\sum_{i=1}^{2} Z_{i2}$	$\sum_{i,j=1}^{2} Z_{ij}$	$\sum_{i=1}^{2} F_i$	TU
Value Added	V_1	V_2	$\sum_{j=1}^{2} V_j$		
Total Output	X_1	X_2	$\sum_{j=1}^{2} X_j$		
Imports	IM_1	IM_2	$\sum_{i=1}^{2} IM_i$		
Total Supply	TS_1	TS_2	TS		

Each row of the SIOTs gives the quantities (in money terms) of each product used in the production of the other products of the economy, while each column gives the quantities of the different products used in the production of each product. Thus, Z_{ij} denotes the quantity of product i used in the production of product j. The column of final uses gives the uses of products for final

[7] What follows can easily be extended in the case of industry-by-industry SIOTs without affecting our main arguments.

consumption, gross capital formation and exports, while the row of value added gives the components of value added per homogeneous unit of production, i.e. compensation of employees, other net taxes on production, consumption of fixed capital and net operating surplus. The last column of the SIOTs gives the total uses (intermediate consumption plus final uses) by product, while the last row of the table shows the total supply in the economy, i.e. total output per product plus imports. Thus, F_i denotes the quantity of product i used for final demand; TU_i the total use of product i in the economy; V_j the value added in the production of product j; X_j the total output of product j; IM_i the imports of product i; TS_i the total supply of product i; TU the total uses in the economy; and TS the total supply in the economy (all data are expressed in monetary units).

By construction of the SIOTs, the following identities hold

$$\sum_{j=1}^{2} Z_{ij} + F_i \equiv TU_i \tag{1.13}$$

$$\sum_{i=1}^{2} TU_i \equiv TU \tag{1.14}$$

$$\sum_{i=1}^{2} Z_{ij} + V_j \equiv X_j \tag{1.15}$$

$$TS_i \equiv X_j + IM_i,\ \forall\, j = i \tag{1.16}$$

$$\sum_{i=1}^{2} TS_i \equiv TS \tag{1.17}$$

$$TS_i \equiv TU_i \tag{1.18}$$

From identities (1.14), (1.17) and (1.18), it follows that

$$TS \equiv TU$$

From identities (1.13), (1.16) and (1.18), it also follows that

$$X_i + IM_i \equiv \sum_{j=1}^{2} Z_{ij} + F_i \tag{1.19}$$

The identities (1.15) and (1.19) can be re-written in vector–matrix terms as follows[8]

$$\mathbf{x}^{\mathrm{T}} \equiv \mathbf{e}^{\mathrm{T}}\mathbf{Z} + \mathbf{v}^{\mathrm{T}}$$

$$\mathbf{x} + \mathbf{Im} \equiv \mathbf{Ze} + \mathbf{f}$$

[8] The transpose of an $nx1$ vector $\boldsymbol{\chi} \equiv [\chi_i]$ is denoted by $\boldsymbol{\chi}^{\mathrm{T}}$, the diagonal matrix formed from the elements of $\boldsymbol{\chi}$ is denoted by $\hat{\chi}$, and $\mathbf{e}$ denotes the summation vector, i.e. $\mathbf{e} \equiv [1,1,\ldots,1]^{\mathrm{T}}$.

where $\mathbf{x} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ is the total output vector; and $\mathbf{Z} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}$ is the matrix of intermediate inputs.

From the previous, it follows that the SIOTs describe an economy in which each production process produces only one product and each product is produced by only one production process. Thus, the SIOTs correspond to an economic world of single production. As it has already been mentioned, those tables of single production are derived from the SUTs, which allow for joint production activities, on the basis of specific assumptions on the relationships between inputs and outputs. The alternative assumptions that have been proposed to convert SUTs into SIOTs are discussed in the next section.

1.4 Conversion of Supply and Use Tables into Symmetric Input–Output Tables

1.4.1 Methods Converting Supply and Use Tables into Product-by-Product Symmetric Input–Output Tables

Since the introduction of SUTs to the System of National Accounts, there has been an ongoing discussion on how these tables can be converted into single production tables (i.e. SIOTs).[9] Most of the discussion has been focused on two alternative assumptions for dealing with the problem at hand: (i) the Product Technology Assumption (PTA hereafter); and (ii) the Industry Technology Assumption (ITA hereafter).

Leaving aside, for simplicity's sake, the imports of the economy, then, the SUTs described by identities (1.11) and (1.12) reduces to the following equations

$$\mathbf{Me} = \mathbf{Ue} + \mathbf{f} \tag{1.20}$$

$$\mathbf{e}^{\mathrm{T}}\mathbf{M} = \mathbf{e}^{\mathrm{T}}\mathbf{U} + \mathbf{v}^{\mathrm{T}} \tag{1.21}$$

The conversion methods try to transform Eqs. 1.20 and 1.21 into a single production system (SIOTs) described by the following equations

$$\mathbf{x} = \mathbf{Ze} + \mathbf{f}^{*} \tag{1.22}$$

$$\mathbf{x}^{\mathrm{T}} = \mathbf{e}^{\mathrm{T}}\mathbf{Z} + \mathbf{v}^{*\mathrm{T}} \tag{1.23}$$

where $\mathbf{Z} \equiv [\mathbf{A}(\mathbf{U},\mathbf{M})\mathbf{x}]$, $\mathbf{A}(\mathbf{U},\mathbf{M})$ is the technical coefficients matrix that is derived from the conversion of SUTs into SIOTs, and $\mathbf{f}^{*}$, $\mathbf{v}^{*\mathrm{T}}$ are the transformed vectors of final demand and value added, respectively. It should be clear that the

[9] What follows draws on Mariolis (2008); Soklis (2009; 2012, Chap. 4).

conversion of SUTs into SIOTs is meaningful only in the case where the make matrix, $\mathbf{M}$, is non-diagonal, i.e. there are industries with secondary production. In the case where there is no secondary production in the economy, i.e. the matrix $\mathbf{M}$ is diagonal, then the system of Eqs. 1.20 and 1.21 is equivalent to the system of Eqs. 1.22 and 1.23. Hence, the purpose of the conversion methods presented below is, by definition, the elimination of secondary production in the economic system.

1.4.1.1 The Product Technology Assumption

The PTA assumes that each industry produces only the total output of the product that is primary to that industry and that each product has its own input structure, irrespective of the industry that produces it. In formal terms, it is assumed that (see, e.g. van Rijckeghem 1967; United Nations 1968, p. 49)[10]

$$\mathbf{A}(\mathbf{U},\mathbf{M}) = \mathbf{U}[\widehat{\mathbf{e}^{\mathrm{T}}\mathbf{M}}]^{-1}[\widehat{\mathbf{e}^{\mathrm{T}}\mathbf{M}}]\mathbf{M}^{-1} = \mathbf{U}\mathbf{M}^{-1} \tag{1.24}$$

and

$$\mathbf{x} = \mathbf{M}\mathbf{e} \tag{1.25}$$

Equation 1.24 entails that

$$\mathbf{U} = \mathbf{A}(\mathbf{U},\mathbf{M})\mathbf{M} \tag{1.26}$$

Post-multiplying Eq. 1.26 by the summation vector gives

$$\mathbf{U}\mathbf{e} = \mathbf{A}(\mathbf{U},\mathbf{M})\mathbf{M}\mathbf{e} \tag{1.27}$$

Substituting Eqs. 1.25 and 1.27 into Eq. 1.20 yields

$$\mathbf{x} = \mathbf{A}(\mathbf{U},\mathbf{M})\mathbf{x} + \mathbf{f} \tag{1.28}$$

Thus, from Eqs. 1.28 and 1.22 it follows that $\mathbf{f}^{*} = \mathbf{f}$. From Eqs. 1.26 and 1.21 it follows that

$$\mathbf{e}^{\mathrm{T}}\mathbf{M} = \mathbf{e}^{\mathrm{T}}\mathbf{A}(\mathbf{U},\mathbf{M})\mathbf{M} + \mathbf{v}^{\mathrm{T}} \tag{1.29}$$

Post-multiplying Eq. 1.29 by $\mathbf{M}^{-1}[\widehat{\mathbf{M}\mathbf{e}}]$ gives

$$\mathbf{e}^{\mathrm{T}}[\widehat{\mathbf{M}\mathbf{e}}] = \mathbf{e}^{\mathrm{T}}\mathbf{A}(\mathbf{U},\mathbf{M})[\widehat{\mathbf{M}\mathbf{e}}] + \mathbf{v}^{\mathrm{T}}\mathbf{M}^{-1}[\widehat{\mathbf{M}\mathbf{e}}] \tag{1.30}$$

Substituting Eq. 1.25 into Eq. 1.30 we obtain

$$\mathbf{x}^{\mathrm{T}} = \mathbf{e}^{\mathrm{T}}\mathbf{A}(\mathbf{U},\mathbf{M})\hat{\mathbf{x}} + \mathbf{v}^{\mathrm{T}}\mathbf{M}^{-1}\hat{\mathbf{x}} \tag{1.31}$$

From Eqs. 1.31, 1.23 and 1.25 it follows that $\mathbf{v}^{*\mathrm{T}} = \mathbf{v}^{\mathrm{T}}\mathbf{M}^{-1}[\widehat{\mathbf{M}\mathbf{e}}]$. Hence, under the PTA, the joint production system described by Eqs. 1.20 and 1.21 is converted into the single production system described by the following equations

$$\mathbf{M}\mathbf{e} = \mathbf{A}(\mathbf{U},\mathbf{M})\mathbf{M}\mathbf{e} + \mathbf{f} \tag{1.32}$$

and

[10] The origins of this method can be found in Edmonston (1952, p. 567).

$$[\mathbf{Me}]^{\mathrm{T}} = \mathbf{e}^{\mathrm{T}}\mathbf{A}(\mathbf{U},\mathbf{M})[\widehat{\mathbf{Me}}] + \mathbf{v}^{\mathrm{T}}\mathbf{M}^{-1}[\widehat{\mathbf{Me}}] \tag{1.33}$$

1.4.1.2 The Industry Technology Assumption

The ITA assumes that each industry produces only the total output of the product that is primary to that industry and that has the same input requirements for any unit of output. In that case, the input structure of each product depends on what industry produces it. In formal terms, it is assumed that (see, e.g. United Nations 1968, pp. 49–50)

$$\mathbf{A}(\mathbf{U},\mathbf{M}) = \mathbf{U}[\widehat{\mathbf{e}^{\mathrm{T}}\mathbf{M}}]^{-1}\mathbf{M}^{\mathrm{T}}[\widehat{\mathbf{Me}}]^{-1} \tag{1.34}$$

and

$$\mathbf{x} = \mathbf{Me} \tag{1.35}$$

Equation 1.34 entails that

$$\mathbf{U} = \mathbf{A}(\mathbf{U},\mathbf{M})[\widehat{\mathbf{Me}}][\mathbf{M}^{\mathrm{T}}]^{-1}[\widehat{\mathbf{e}^{\mathrm{T}}\mathbf{M}}] \tag{1.36}$$

Post-multiplying Eq. 1.36 by the summation vector gives

$$\mathbf{Ue} = \mathbf{A}(\mathbf{U},\mathbf{M})\mathbf{Me} \tag{1.37}$$

Substituting Eqs. 1.35 and 1.37 into Eq. 1.20 yields

$$\mathbf{x} = \mathbf{A}(\mathbf{U},\mathbf{M})\mathbf{x} + \mathbf{f} \tag{1.38}$$

From Eqs. 1.38 and 1.22 it follows that $\mathbf{f}^* = \mathbf{f}$. Substituting Eq. 1.36 into Eq. 1.21 yields

$$\mathbf{e}^{\mathrm{T}}\mathbf{M} = \mathbf{e}^{\mathrm{T}}\mathbf{A}(\mathbf{U},\mathbf{M})[\widehat{\mathbf{Me}}][\mathbf{M}^{\mathrm{T}}]^{-1}[\widehat{\mathbf{e}^{\mathrm{T}}\mathbf{M}}] + \mathbf{v}^{\mathrm{T}} \tag{1.39}$$

Post-multiplying Eq. 1.39 by $[\widehat{\mathbf{e}^{\mathrm{T}}\mathbf{M}}]^{-1}\mathbf{M}^{\mathrm{T}}$ gives

$$\mathbf{e}^{\mathrm{T}}\mathbf{M}[\widehat{\mathbf{e}^{\mathrm{T}}\mathbf{M}}]^{-1}\mathbf{M}^{\mathrm{T}} = \mathbf{e}^{\mathrm{T}}\mathbf{A}(\mathbf{U},\mathbf{M})[\widehat{\mathbf{Me}}] + \mathbf{v}^{\mathrm{T}}[\widehat{\mathbf{e}^{\mathrm{T}}\mathbf{M}}]^{-1}\mathbf{M}^{\mathrm{T}} \tag{1.40}$$

Substituting Eq. 1.35 into Eq. 1.40 we obtain[11]

$$\mathbf{x}^{\mathrm{T}} = \mathbf{e}^{\mathrm{T}}\mathbf{A}(\mathbf{U},\mathbf{M})\hat{\mathbf{x}} + \mathbf{v}^{\mathrm{T}}[\widehat{\mathbf{e}^{\mathrm{T}}\mathbf{M}}]^{-1}\mathbf{M}^{\mathrm{T}} \tag{1.41}$$

From Eqs. 1.41 and 1.23 it follows that $\mathbf{v}^{*\mathrm{T}} = \mathbf{v}^{\mathrm{T}}[\widehat{\mathbf{e}^{\mathrm{T}}\mathbf{M}}]^{-1}\mathbf{M}^{\mathrm{T}}$. Hence, under the ITA, the joint production system described by Eqs. 1.20 and 1.21 is converted into the single production system described by the following equations

$$\mathbf{Me} = \mathbf{A}(\mathbf{U},\mathbf{M})\mathbf{Me} + \mathbf{f} \tag{1.42}$$

and

$$[\mathbf{Me}]^{\mathrm{T}} = \mathbf{e}^{\mathrm{T}}\mathbf{A}(\mathbf{U},\mathbf{M})[\widehat{\mathbf{Me}}] + \mathbf{v}^{\mathrm{T}}[\widehat{\mathbf{e}^{\mathrm{T}}\mathbf{M}}]^{-1}\mathbf{M}^{\mathrm{T}} \tag{1.43}$$

1.4.1.3 The By-Product Method

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[11] Note that $\mathbf{e}^{\mathrm{T}}\mathbf{M}(\widehat{\mathbf{e}^{\mathrm{T}}\mathbf{M}})^{-1}\mathbf{M}^{\mathrm{T}} = \mathbf{e}^{\mathrm{T}}\mathbf{M}^{\mathrm{T}} = (\mathbf{Me})^{\mathrm{T}}$.

The by-product method (Stone 1961, pp. 39–41) assumes that all secondary products are "by-products" and that can be treated as negative inputs of the industries that they are actually produced.[12] The make matrix splits into $\mathbf{M}_1$ and $\mathbf{M}_2$, where $\mathbf{M}_1$ is the diagonal matrix that describes the primary products of each industry and $\mathbf{M}_2$ is the off-diagonal matrix that describes the secondary products of each industry. Thus, it holds

$$\mathbf{M} \equiv \mathbf{M}_1 + \mathbf{M}_2 \tag{1.44}$$

In mathematical terms, the by-product method assumes that (see, e.g. ten Raa et al. 1984, p. 88)

$$\mathbf{A}(\mathbf{U},\mathbf{M}) = [\mathbf{U} - \mathbf{M}_2]\mathbf{M}_1^{-1} \tag{1.45}$$

and

$$\mathbf{x} = \mathbf{M}_1\mathbf{e} \tag{1.46}$$

Equation 1.45 entails that

$$\mathbf{U} - \mathbf{M}_2 = \mathbf{A}(\mathbf{U},\mathbf{M})\mathbf{M}_1 \tag{1.47}$$

Combining Eqs. 1.44 and 1.20 gives

$$\mathbf{M}_1\mathbf{e} = (\mathbf{U} - \mathbf{M}_2)\mathbf{e} + \mathbf{f} \tag{1.48}$$

Substituting Eqs. 1.46 and 1.47 into Eq. 1.48 yields

$$\mathbf{x} = \mathbf{A}(\mathbf{U},\mathbf{M})\mathbf{x} + \mathbf{f} \tag{1.49}$$

From Eqs. 1.22 and 1.49 it follows that $\mathbf{f}^* = \mathbf{f}$. Combining Eqs. 1.44 and 1.21 gives

$$\mathbf{e}^{\mathrm{T}}\mathbf{M}_1 = \mathbf{e}^{\mathrm{T}}[\mathbf{U} - \mathbf{M}_2] + \mathbf{v}^{\mathrm{T}} \tag{1.50}$$

Substituting Eq. 1.45 into Eq. 1.50 yields

$$\mathbf{e}^{\mathrm{T}}\mathbf{M}_1 = \mathbf{e}^{\mathrm{T}}\mathbf{A}(\mathbf{U},\mathbf{M})\mathbf{M}_1 + \mathbf{v}^{\mathrm{T}} \tag{1.51}$$

Substituting Eq. 1.46 into Eq. 1.51 yields[13]

$$\mathbf{x}^{\mathrm{T}} = \mathbf{e}^{\mathrm{T}}\mathbf{A}(\mathbf{U},\mathbf{M})\hat{\mathbf{x}} + \mathbf{v}^{\mathrm{T}} \tag{1.52}$$

From Eqs. 1.52 and 1.23 it follows that $\mathbf{v}^{*\mathrm{T}} = \mathbf{v}^{\mathrm{T}}$. Thus, it follows that, under the by-product method, the joint production system described by Eqs. 1.20 and 1.21 is converted into the single production system described by the equations

$$\mathbf{M}_1\mathbf{e} = \mathbf{A}(\mathbf{U},\mathbf{M})\mathbf{M}_1\mathbf{e} + \mathbf{f}$$

and

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[12] By-products are defined as products that are technologically linked to the production of the primary product of the industry where it is actually produced (Stone 1961, p. 39). The inputs needed for their production are considered to be "low" in relation to the primary product of the industry where they are produced (United Nations 1999, p. 77; Viet 1994).

[13] Note that $\mathbf{e}^{\mathrm{T}}\mathbf{M}_1 = [\mathbf{M}_1\mathbf{e}]^{\mathrm{T}}$ and $\mathbf{M}_1 = [\widehat{\mathbf{M}_1\mathbf{e}}]$.

$$[\mathbf{M}_1\mathbf{e}]^{\mathrm{T}} = \mathbf{e}^{\mathrm{T}}\mathbf{A}(\mathbf{U},\mathbf{M})\mathbf{M}_1 + \mathbf{v}^{\mathrm{T}}$$

1.4.1.4 Mixed Technology Assumptions

Mixed Technology Assumptions were suggested by Gigantes and Matuszweski (1968) and were incorporated in the 1968 System of National Accounts (United Nations 1968, p. 50). This conversion method assumes that a part of the secondary products should be treated using the PTA and the remaining part should be treated using the ITA. The make matrix splits into the matrices $\mathbf{M}_1$ and $\mathbf{M}_2$ and it is assumed that $\mathbf{M}_1$ includes output that fits the PTA whereas $\mathbf{M}_2$ includes output that fits the ITA. Thus, it holds

$$\mathbf{M} \equiv \mathbf{M}_1 + \mathbf{M}_2 \tag{1.53}$$

Following Armstrong (1975), we assume that

$$\mathbf{A}_1 = \mathbf{U}[\widehat{\mathbf{M}^{\mathrm{T}}\mathbf{e}}]^{-1}[\widehat{\mathbf{M}_1^{\mathrm{T}}\mathbf{e}}]\mathbf{M}_1^{-1}$$

and

$$\mathbf{A}_2 = \mathbf{U}[\widehat{\mathbf{M}^{\mathrm{T}}\mathbf{e}}]^{-1}\mathbf{M}_2^{\mathrm{T}}[\widehat{\mathbf{M}_2\mathbf{e}}]^{-1}$$

where $\mathbf{A}_1$ is the technical coefficients matrix for outputs included in $\mathbf{M}_1$, while $\mathbf{A}_2$ is the technical coefficients matrix for outputs included in $\mathbf{M}_2$. Then, the technical coefficients matrix of the economy is given by[14]

$$\mathbf{A}(\mathbf{U},\mathbf{M}) = \mathbf{A}_1[\mathbf{I} - \mathbf{D}] + \mathbf{A}_2\mathbf{D}$$

where $\mathbf{D} \equiv [\widehat{\mathbf{M}_2\mathbf{e}}][\widehat{\mathbf{M}\mathbf{e}}]^{-1}$. It can be seen that if $\mathbf{M}_2 = \mathbf{0}$, then $\mathbf{A}(\mathbf{U},\mathbf{M}) = \mathbf{A}_1 = \mathbf{U}\mathbf{M}^{-1}$, which is the PTA. On the other hand, if $\mathbf{M}_1 = \mathbf{0}$, then $\mathbf{A}(\mathbf{U},\mathbf{M}) = \mathbf{A}_2 = \mathbf{U}[\widehat{\mathbf{e}^{\mathrm{T}}\mathbf{M}}]^{-1}\mathbf{M}^{\mathrm{T}}[\widehat{\mathbf{M}\mathbf{e}}]^{-1}$, which is the ITA.

On the basis of their critique to the ITA, ten Raa et al. (1984) constructed an alternative mixed technology model, assuming that $\mathbf{M}_2$ includes output that fits the by-product assumption. In mathematical terms, it is assumed that

$$\mathbf{A}(\mathbf{U},\mathbf{M}) = [\mathbf{U} - \mathbf{M}_2]\mathbf{M}_1^{-1} \tag{1.54}$$

and

$$\mathbf{x} = \mathbf{M}_1\mathbf{e} \tag{1.55}$$

Equation 1.54 entails that

$$\mathbf{U} - \mathbf{M}_2 = \mathbf{A}(\mathbf{U},\mathbf{M})\mathbf{M}_1 \tag{1.56}$$

Combining Eqs. 1.53 and 1.20 gives

[14] For alternative ways of calculating the technical coefficients matrix, using mixed technology assumptions, see Armstrong (1975, pp. 74–76); Gigantes (1970, pp. 284–290).

$$\mathbf{M}_1\mathbf{e} = [\mathbf{U} - \mathbf{M}_2]\mathbf{e} + \mathbf{f} \tag{1.57}$$

Substituting Eqs. 1.55 and 1.56 into Eq. 1.57 yields

$$\mathbf{x} = \mathbf{A}(\mathbf{U},\mathbf{M})\mathbf{x} + \mathbf{f} \tag{1.58}$$

From Eqs. 1.58 and 1.22 it follows that $\mathbf{f}^* = \mathbf{f}$. Combining Eqs. 1.53 and 1.21 gives

$$\mathbf{e}^{\mathrm{T}}\mathbf{M}_1 = \mathbf{e}^{\mathrm{T}}[\mathbf{U} - \mathbf{M}_2] + \mathbf{v}^{\mathrm{T}} \tag{1.59}$$

Substituting Eq. 1.56 into Eq. 1.59 yields

$$\mathbf{e}^{\mathrm{T}}\mathbf{M}_1 = \mathbf{e}^{\mathrm{T}}\mathbf{A}(\mathbf{U},\mathbf{M})\mathbf{M}_1 + \mathbf{v}^{\mathrm{T}} \tag{1.60}$$

Post-multiplying Eq. 1.60 by $\mathbf{M}_1^{-1}[\widehat{\mathbf{M}_1\mathbf{e}}]$ we obtain

$$\mathbf{e}^{\mathrm{T}}[\widehat{\mathbf{M}_1\mathbf{e}}] = \mathbf{e}^{\mathrm{T}}\mathbf{A}(\mathbf{U},\mathbf{M})[\widehat{\mathbf{M}_1\mathbf{e}}] + \mathbf{v}^{\mathrm{T}}\mathbf{M}_1^{-1}[\widehat{\mathbf{M}_1\mathbf{e}}] \tag{1.61}$$

Substituting Eq. 1.55 into Eq. 1.61 yields

$$\mathbf{x}^{\mathrm{T}} = \mathbf{e}^{\mathrm{T}}\mathbf{A}(\mathbf{U},\mathbf{M})\hat{\mathbf{x}} + \mathbf{v}^{\mathrm{T}}\mathbf{M}_1^{-1}\hat{\mathbf{x}} \tag{1.62}$$

From Eqs. 1.62, 1.55 and 1.23 it follows that $\mathbf{v}^{*\mathrm{T}} = \mathbf{v}^{\mathrm{T}}\mathbf{M}_1^{-1}[\widehat{\mathbf{M}_1\mathbf{e}}]$. Thus, it follows that, under the mixed technology model introduced by ten Raa et al. (1984), the joint production system described by Eqs. 1.20 and 1.21 is converted into the single production system described by

$$\mathbf{M}_1\mathbf{e} = \mathbf{A}(\mathbf{U},\mathbf{M})\mathbf{M}_1\mathbf{e} + \mathbf{f}$$

and

$$[\mathbf{M}_1\mathbf{e}]^{\mathrm{T}} = \mathbf{e}^{\mathrm{T}}\mathbf{A}(\mathbf{U},\mathbf{M})[\widehat{\mathbf{M}_1\mathbf{e}}] + \mathbf{v}^{\mathrm{T}}\mathbf{M}_1^{-1}[\widehat{\mathbf{M}_1\mathbf{e}}]$$

1.4.1.5 The Transfer Method

The Transfer method was proposed by Stone (1961, pp. 39–41) as an alternative method to treat by-products. This method treats secondary products as if they were bought by the industry where they are "primary" and added to the output of that industry. In formal terms, it is assumed that (see, e.g. Jansen and ten Raa 1990, p. 215)

$$\mathbf{x} = [\mathbf{M} + \mathbf{M}_2^{\mathrm{T}}]\mathbf{e} \tag{1.63}$$

and

$$\mathbf{A}(\mathbf{U},\mathbf{M}) = [\mathbf{U} + \mathbf{M}_2^{\mathrm{T}}][[\widehat{\mathbf{e}^{\mathrm{T}}\mathbf{M}}] + [\widehat{\mathbf{M}\mathbf{e}}] - \mathbf{M}_1]^{-1} \tag{1.64}$$

where $\mathbf{M}_1$ is the diagonal matrix that describes the primary products of each industry and $\mathbf{M}_2$ is the off-diagonal matrix that describes the secondary products of each industry. Equation 1.64 entails that

$$\mathbf{U} + \mathbf{M}_2^{\mathrm{T}} = \mathbf{A}(\mathbf{U},\mathbf{M})[[\widehat{\mathbf{e}^{\mathrm{T}}\mathbf{M}}] + [\widehat{\mathbf{M}\mathbf{e}}] - \mathbf{M}_1] \tag{1.65}$$

Adding $\mathbf{M}_2^{\mathrm{T}}\mathbf{e}$ to both sides of Eq. 1.20 we obtain

$$[\mathbf{M}+\mathbf{M}_2^{\mathrm{T}}]\mathbf{e}=[\mathbf{U}+\mathbf{M}_2^{\mathrm{T}}]\mathbf{e}+\mathbf{f} \tag{1.66}$$

Substituting Eq. 1.65 into Eq. 1.66 yields

$$[\mathbf{M}+\mathbf{M}_2^{\mathrm{T}}]\mathbf{e}=\mathbf{A}(\mathbf{U},\mathbf{M})[\widehat{[\mathbf{e}^{\mathrm{T}}\mathbf{M}]}+\widehat{[\mathbf{Me}]}-\mathbf{M}_1]\mathbf{e}+\mathbf{f}$$

or[15]

$$[\mathbf{M}+\mathbf{M}_2^{\mathrm{T}}]\mathbf{e}=\mathbf{A}(\mathbf{U},\mathbf{M})[\mathbf{M}+\mathbf{M}_2^{\mathrm{T}}]\mathbf{e}+\mathbf{f} \tag{1.67}$$

Substituting Eq. 1.63 into Eq. 1.67 gives

$$\mathbf{x}=\mathbf{A}(\mathbf{U},\mathbf{M})\mathbf{x}+\mathbf{f} \tag{1.68}$$

From Eqs. 1.68 and 1.22 it follows that $\mathbf{f}^{*}=\mathbf{f}$. Adding $\mathbf{e}^{\mathrm{T}}\mathbf{M}_2^{\mathrm{T}}$ to both sides of Eq. 1.21 we obtain

$$\mathbf{e}^{\mathrm{T}}[\mathbf{M}+\mathbf{M}_2^{\mathrm{T}}]=\mathbf{e}^{\mathrm{T}}[\mathbf{U}+\mathbf{M}_2^{\mathrm{T}}]+\mathbf{v}^{\mathrm{T}} \tag{1.69}$$

Substituting Eq. 1.65 into Eq. 1.69 yields

$$\mathbf{e}^{\mathrm{T}}[\mathbf{M}+\mathbf{M}_2^{\mathrm{T}}]=\mathbf{e}^{\mathrm{T}}\mathbf{A}(\mathbf{U},\mathbf{M})[\widehat{[\mathbf{e}^{\mathrm{T}}\mathbf{M}]}+\widehat{[\mathbf{Me}]}-\mathbf{M}_1]+\mathbf{v}^{\mathrm{T}}$$

or

$$\mathbf{x}^{\mathrm{T}}=\mathbf{e}^{\mathrm{T}}\mathbf{A}(\mathbf{U},\mathbf{M})\hat{\mathbf{x}}+\mathbf{v}^{\mathrm{T}} \tag{1.70}$$

From Eqs. 1.70 and 1.23 it follows that $\mathbf{v}^{*\mathrm{T}}=\mathbf{v}^{\mathrm{T}}$. Hence, it follows that, under the transfer method, the joint production system described by Eqs. 1.20 and 1.21 is converted into the single production system described by

$$[\mathbf{M}+\mathbf{M}_2^{\mathrm{T}}]\mathbf{e}=\mathbf{A}(\mathbf{U},\mathbf{M})[\mathbf{M}+\mathbf{M}_2^{\mathrm{T}}]\mathbf{e}+\mathbf{f}$$

and

$$[[\mathbf{M}+\mathbf{M}_2^{\mathrm{T}}]\mathbf{e}]^{\mathrm{T}}=\mathbf{e}^{\mathrm{T}}\mathbf{A}(\mathbf{U},\mathbf{M})[\widehat{[\mathbf{M}+\mathbf{M}_2^{\mathrm{T}}]\mathbf{e}}]+\mathbf{v}^{\mathrm{T}}$$

1.4.1.6 The ESA Method

The ESA (European System of Integrated Economic Accounts) method (Eurostat 1979, pp. 116–117) recommends that secondary products should be treated as if they were produced by the industries were these products are primary. In formal terms, it is assumed that (see, e.g. Viet 1994, pp. 38–40)

$$\mathbf{x}=\mathbf{Me} \tag{1.71}$$

and

$$\mathbf{A}(\mathbf{U},\mathbf{M})=\mathbf{U}\widehat{[\mathbf{Me}]}^{-1} \tag{1.72}$$

Equation 1.72 entails that

$$\mathbf{U}=\mathbf{A}(\mathbf{U},\mathbf{M})\widehat{[\mathbf{Me}]} \tag{1.73}$$

Substituting Eq. 1.73 into Eq. 1.20 yields

[15] Note that $\widehat{[\mathbf{e}^{\mathrm{T}}\mathbf{M}]}+\widehat{[\mathbf{Me}]}-\mathbf{M}_1=\widehat{[\mathbf{M}+\mathbf{M}_2^{\mathrm{T}}]\mathbf{e}}$.

$$\mathbf{Me} = \mathbf{A}(\mathbf{U},\mathbf{M})\mathbf{Me} + \mathbf{f} \tag{1.74}$$

Substituting Eq. 1.71 into Eq. 1.74 gives

$$\mathbf{x} = \mathbf{A}(\mathbf{U},\mathbf{M})\mathbf{x} + \mathbf{f} \tag{1.75}$$

From Eqs. 1.75 and 1.22 it follows that $\mathbf{f}^* = \mathbf{f}$. Substituting Eq. 1.73 into Eq. 1.21 yields

$$\mathbf{e}^{\mathrm{T}}\mathbf{M} = \mathbf{e}^{\mathrm{T}}\mathbf{A}(\mathbf{U},\mathbf{M})[\widehat{\mathbf{Me}}] + \mathbf{v}^{\mathrm{T}} \tag{1.76}$$

Adding $[\mathbf{Me}]^{\mathrm{T}}$ to both sides of Eq. 1.76 and after rearrangement we obtain

$$[\mathbf{Me}]^{\mathrm{T}} = \mathbf{e}^{\mathrm{T}}\mathbf{A}(\mathbf{U},\mathbf{M})[\widehat{\mathbf{Me}}] + \mathbf{v}^{\mathrm{T}} + [\mathbf{Me}]^{\mathrm{T}} - \mathbf{e}^{\mathrm{T}}\mathbf{M} \tag{1.77}$$

Substituting Eq. 1.71 into Eq. 1.77 yields

$$\mathbf{x}^{\mathrm{T}} = \mathbf{e}^{\mathrm{T}}\mathbf{A}(\mathbf{U},\mathbf{M})\hat{\mathbf{x}} + \mathbf{v}^{\mathrm{T}} + [\mathbf{Me}]^{\mathrm{T}} - \mathbf{e}^{\mathrm{T}}\mathbf{M} \tag{1.78}$$

From Eqs. 1.78 and 1.23 it follows that $\mathbf{v}^{*\mathrm{T}} = \mathbf{v}^{\mathrm{T}} + [\mathbf{Me}]^{\mathrm{T}} - \mathbf{e}^{\mathrm{T}}\mathbf{M}$. Consequently, with the use of the ESA method, the joint production system described by Eqs. 1.20 and 1.21 is converted into the single production system described by the equations

$$\mathbf{Me} = \mathbf{A}(\mathbf{U},\mathbf{M})\mathbf{Me} + \mathbf{f}$$

and

$$[\mathbf{Me}]^{\mathrm{T}} = \mathbf{e}^{\mathrm{T}}\mathbf{A}(\mathbf{U},\mathbf{M})[\widehat{\mathbf{Me}}] + \mathbf{v}^{\mathrm{T}} + [\mathbf{Me}]^{\mathrm{T}} - \mathbf{e}^{\mathrm{T}}\mathbf{M}$$

1.4.1.7 The Lump-Sum Method

The Lump-Sum (or Aggregation) Method (Office of Statistical Standards 1974, p. 116) treats secondary products as if they were produced as a primary product of the industry that they are actually produced. In formal terms, it is assumed that (see, e.g. Fukui and Seneta 1985, p. 177)

$$\mathbf{x} = \mathbf{M}^{\mathrm{T}}\mathbf{e} \tag{1.79}$$

and

$$\mathbf{A}(\mathbf{U},\mathbf{M}) = \mathbf{U}[\widehat{\mathbf{M}^{\mathrm{T}}\mathbf{e}}]^{-1} \tag{1.80}$$

Equation 1.80 entails that

$$\mathbf{U} = \mathbf{A}(\mathbf{U},\mathbf{M})[\widehat{\mathbf{M}^{\mathrm{T}}\mathbf{e}}] \tag{1.81}$$

Substituting Eq. 1.81 into Eq. 1.20 yields

$$\mathbf{Me} = \mathbf{A}(\mathbf{U},\mathbf{M})\mathbf{M}^{\mathrm{T}}\mathbf{e} + \mathbf{f} \tag{1.82}$$

Adding $\mathbf{M}^{\mathrm{T}}\mathbf{e}$ to both sides of Eq. 1.82 and after rearrangement we obtain

$$\mathbf{M}^{\mathrm{T}}\mathbf{e} = \mathbf{A}(\mathbf{U},\mathbf{M})\mathbf{M}^{\mathrm{T}}\mathbf{e} + \mathbf{f} + [\mathbf{M}^{\mathrm{T}} - \mathbf{M}]\mathbf{e} \tag{1.83}$$

Substituting Eq. 1.79 into Eq. 1.83 yields

$$\mathbf{x} = \mathbf{A}(\mathbf{U},\mathbf{M})\mathbf{x} + \mathbf{f} + [\mathbf{M}^{\mathrm{T}} - \mathbf{M}]\mathbf{e} \tag{1.84}$$

From Eqs. 1.84 and 1.22 it follows that $\mathbf{f}^{*} = \mathbf{f} + [\mathbf{M}^{\mathrm{T}} - \mathbf{M}]\mathbf{e}$. Substituting Eq. 1.81 into Eq. 1.21 gives

$$\mathbf{e}^{\mathrm{T}}\mathbf{M} = \mathbf{e}^{\mathrm{T}}\mathbf{A}(\mathbf{U},\mathbf{M})[\widehat{\mathbf{M}^{\mathrm{T}}\mathbf{e}}] + \mathbf{v}^{\mathrm{T}} \tag{1.85}$$

Combining Eqs. 1.79 and 1.85 yields[16]

$$\mathbf{x}^{\mathrm{T}} = \mathbf{e}^{\mathrm{T}}\mathbf{A}(\mathbf{U},\mathbf{M})\hat{\mathbf{x}} + \mathbf{v}^{\mathrm{T}} \tag{1.86}$$

From Eqs. 1.86 and 1.23 it follows that $\mathbf{v}^{*\mathrm{T}} = \mathbf{v}^{\mathrm{T}}$. Hence, under the Lump-Sum Method, the joint production system described by Eqs. 1.20 and 1.21 is converted into the single production system described by the equations

$$\mathbf{M}^{\mathrm{T}}\mathbf{e} = \mathbf{A}(\mathbf{U},\mathbf{M})\mathbf{M}^{\mathrm{T}}\mathbf{e} + \mathbf{f} + [\mathbf{M}^{\mathrm{T}} - \mathbf{M}]\mathbf{e}$$

and

$$[\mathbf{M}^{\mathrm{T}}\mathbf{e}]^{\mathrm{T}} = \mathbf{e}^{\mathrm{T}}\mathbf{A}(\mathbf{U},\mathbf{M})[\widehat{\mathbf{M}^{\mathrm{T}}\mathbf{e}}] + \mathbf{v}^{\mathrm{T}}$$

1.4.1.8 The Redefinition Method

The Redefinition method is used to move outputs and inputs of secondary products, that have distinctive production processes compared to those of the primary products of each industry, to the industries where these products are primary (see, e.g. Viet 1994, p. 40). This method is most suitable to be applied for secondary products that have production processes similar to the respective production processes of the industries where these products are primary. Nevertheless, this method needs additional data on the production of the secondary products that are not always available (see, e.g. United Nations 1999, p. 81).[17]

1.4.2 Methods Converting Supply and Use Tables into Industry-by-Industry Symmetric Input–Output Tables

The conversion methods presented so far derive SIOTs where the dimensions of the derived matrix of intermediate inputs and, hence, the technical coefficients matrix, $\mathbf{A}(\mathbf{U},\mathbf{M})$, is product-by-product, i.e. represents the transactions amongst the different products of the economy. However, following conjugated procedures with those previously presented we may convert SUTs into SIOTs where the dimensions of the derived matrix of intermediate inputs is industry-by-industry, i.e. represents the transactions amongst the different industries of the economy.

[16] Note that $\mathbf{e}^{\mathrm{T}}\mathbf{M} = (\mathbf{M}^{\mathrm{T}}\mathbf{e})^{\mathrm{T}}$.

[17] For a presentation of the results that the Redefinition method yielded in the case of the USA input-output tables for the year 1992, see Guo et al. (2002, pp. 11–13).

In what follows we present two such conversion methods, known as the Fixed Industry Sales Assumption (FISA hereafter) and Fixed Product Sales Assumption (FPSA hereafter), respectively.

1.4.2.1 The Fixed Industry Sales Assumption

The FISA treats secondary products as if they were produced as a primary product of the industry that they are actually produced and assumes that each industry has its own sales structure, irrespective of its product mix. In formal terms, it is assumed that

$$\mathbf{A}(\mathbf{U},\mathbf{M}) = [\widehat{\mathbf{e}^{\mathrm{T}}\mathbf{M}}]\mathbf{M}^{-1}\mathbf{U}[\widehat{\mathbf{e}^{\mathrm{T}}\mathbf{M}}]^{-1} \tag{1.87}$$

and

$$\mathbf{x} = \mathbf{M}^{\mathrm{T}}\mathbf{e} \tag{1.88}$$

Equation 1.87 entails that

$$\mathbf{U} = \mathbf{M}[\widehat{\mathbf{e}^{\mathrm{T}}\mathbf{M}}]^{-1}\mathbf{A}(\mathbf{U},\mathbf{M})[\widehat{\mathbf{e}^{\mathrm{T}}\mathbf{M}}] \tag{1.89}$$

Substituting Eqs. 1.88 and 1.89 into Eq. 1.21 yields

$$\mathbf{x}^{\mathrm{T}} = \mathbf{e}^{\mathrm{T}}\mathbf{M}[\widehat{\mathbf{e}^{\mathrm{T}}\mathbf{M}}]^{-1}\mathbf{A}(\mathbf{U},\mathbf{M})\mathbf{x} + \mathbf{v}^{\mathrm{T}}$$

or

$$\mathbf{x}^{\mathrm{T}} = \mathbf{e}^{\mathrm{T}}\mathbf{A}(\mathbf{U},\mathbf{M})\hat{\mathbf{x}} + \mathbf{v}^{\mathrm{T}} \tag{1.90}$$

From Eqs. 1.90 and 1.23 it follows that $\mathbf{v}^{*\mathrm{T}} = \mathbf{v}^{\mathrm{T}}$. Substituting Eq. 1.89 into Eq. 1.20 yields

$$\mathbf{M}\mathbf{e} = \mathbf{M}[\widehat{\mathbf{e}^{\mathrm{T}}\mathbf{M}}]^{-1}\mathbf{A}(\mathbf{U},\mathbf{M})[\widehat{\mathbf{e}^{\mathrm{T}}\mathbf{M}}]\mathbf{e} + \mathbf{f} \tag{1.91}$$

Pre-multiplying Eq. 1.91 by $[\widehat{\mathbf{e}^{\mathrm{T}}\mathbf{M}}]\mathbf{M}^{-1}$ and taking into account Eq. 1.88, we obtain

$$\mathbf{x} = \mathbf{A}(\mathbf{U},\mathbf{M})\mathbf{x} + [\widehat{\mathbf{e}^{\mathrm{T}}\mathbf{M}}]\mathbf{M}^{-1}\mathbf{f} \tag{1.92}$$

From Eqs. 1.92 and 1.22 it follows that $\mathbf{f}^{*} = [\widehat{\mathbf{e}^{\mathrm{T}}\mathbf{M}}]\mathbf{M}^{-1}\mathbf{f}$. Hence, under the FISA, the joint production system described by Eqs. 1.20 and 1.21 is converted into the single production system described by the following equations

$$\mathbf{M}^{\mathrm{T}}\mathbf{e} = \mathbf{A}(\mathbf{U},\mathbf{M})\mathbf{M}^{\mathrm{T}}\mathbf{e} + [\widehat{\mathbf{e}^{\mathrm{T}}\mathbf{M}}]\mathbf{M}^{-1}\mathbf{f} \tag{1.93}$$

and

$$\mathbf{e}^{\mathrm{T}}\mathbf{M} = \mathbf{e}^{\mathrm{T}}\mathbf{A}(\mathbf{U},\mathbf{M})[\widehat{\mathbf{e}^{\mathrm{T}}\mathbf{M}}] + \mathbf{v}^{\mathrm{T}} \tag{1.94}$$

Comparing Eqs. 1.32 and 1.33 with Eqs. 1.93 and 1.94, i.e. the single production system derived under the PTA with the single production system derived under the FISA, it follows that

$$\mathbf{A}_{\mathrm{I}}(\mathbf{U},\mathbf{M}) = \mathbf{S}\mathbf{A}_{\mathrm{C}}(\mathbf{U},\mathbf{M})\mathbf{S}^{-1}$$

$$\mathbf{f}_{\mathrm{I}}^{*} = \mathbf{S}\mathbf{f}_{\mathrm{C}}^{*}$$

$$\boldsymbol{\pi}_{\mathrm{I}}^{*\mathrm{T}} = \boldsymbol{\pi}_{\mathrm{C}}^{*\mathrm{T}}\mathbf{S}^{-1}$$

and

$$\mathbf{x}_{\mathrm{I}} = \mathbf{S}\mathbf{x}_{\mathrm{C}}$$

where $\mathbf{S} \equiv [\widehat{\mathbf{e}^{\mathrm{T}}\mathbf{M}}]\mathbf{M}^{-1}$, while $\mathbf{A}_{\mathrm{I}}(\mathbf{U},\mathbf{M})$ ($\mathbf{A}_{\mathrm{C}}(\mathbf{U},\mathbf{M})$), $\mathbf{f}_{\mathrm{I}}^{*}$ ($\mathbf{f}_{\mathrm{C}}^{*}$), $\boldsymbol{\pi}_{\mathrm{I}}^{*\mathrm{T}} \equiv [\mathbf{v}_{\mathrm{I}}^{*\mathrm{T}}\hat{\mathbf{x}}_{\mathrm{I}}^{-1}]$ ($\boldsymbol{\pi}_{\mathrm{C}}^{*\mathrm{T}} \equiv [\mathbf{v}_{\mathrm{C}}^{*\mathrm{T}}\hat{\mathbf{x}}_{\mathrm{C}}^{-1}]$) and $\mathbf{x}_{\mathrm{I}}$ ($\mathbf{x}_{\mathrm{C}}$) is the technical coefficients matrix, the transformed vector of final demand, the transformed vector of value added coefficients and the transformed vector of value added, respectively, derived under the FISA (PTA). These relationships imply that the FISA (PTA) can be derived from PTA (FISA) via a similarity transformation.

1.4.2.2 The Fixed Product Sales Assumption

The FPSA treats secondary products as if they were produced as a primary product of the industry that they are actually produced and assumes that each product has its own market shares independent of the industry where it is produced. In formal terms, it is assumed that

$$\mathbf{A}(\mathbf{U},\mathbf{M}) = \mathbf{M}^{\mathrm{T}}[\widehat{\mathbf{M}\mathbf{e}}]^{-1}\mathbf{U}[\widehat{\mathbf{e}^{\mathrm{T}}\mathbf{M}}]^{-1} \tag{1.95}$$

and

$$\mathbf{x} = \mathbf{M}^{\mathrm{T}}\mathbf{e} \tag{1.96}$$

Equation 1.95 entails that

$$\mathbf{U} = [\widehat{\mathbf{M}\mathbf{e}}][\mathbf{M}^{\mathrm{T}}]^{-1}\mathbf{A}(\mathbf{U},\mathbf{M})[\widehat{\mathbf{e}^{\mathrm{T}}\mathbf{M}}] \tag{1.97}$$

Substituting Eqs. 1.96 and 1.97 into Eq. 1.21 yields

$$\mathbf{x}^{\mathrm{T}} = \mathbf{e}^{\mathrm{T}}[\widehat{\mathbf{M}\mathbf{e}}][\mathbf{M}^{\mathrm{T}}]^{-1}\mathbf{A}(\mathbf{U},\mathbf{M})\mathbf{x} + \mathbf{v}^{\mathrm{T}}$$

or

$$\mathbf{x}^{\mathrm{T}} = \mathbf{e}^{\mathrm{T}}\mathbf{M}^{\mathrm{T}}[\mathbf{M}^{\mathrm{T}}]^{-1}\mathbf{A}(\mathbf{U},\mathbf{M})\hat{\mathbf{x}} + \mathbf{v}^{\mathrm{T}}$$

or

$$\mathbf{x}^{\mathrm{T}} = \mathbf{e}^{\mathrm{T}}\mathbf{A}(\mathbf{U},\mathbf{M})\hat{\mathbf{x}} + \mathbf{v}^{\mathrm{T}} \tag{1.98}$$

From Eqs. 1.98 and 1.23 it follows that $\mathbf{v}^{*\mathrm{T}} = \mathbf{v}^{\mathrm{T}}$. Substituting Eq. 1.97 into Eq. 1.20 yields

$$\mathbf{M}\mathbf{e} = [\widehat{\mathbf{M}\mathbf{e}}][\mathbf{M}^{\mathrm{T}}]^{-1}\mathbf{A}(\mathbf{U},\mathbf{M})[\widehat{\mathbf{e}^{\mathrm{T}}\mathbf{M}}]\mathbf{e} + \mathbf{f} \tag{1.99}$$

Pre-multiplying Eq. 1.99 by $\mathbf{M}^{\mathrm{T}}[\widehat{\mathbf{M}\mathbf{e}}]^{-1}$ and taking into account Eq. 1.96, we obtain

$$\mathbf{x} = \mathbf{A}(\mathbf{U},\mathbf{M})\mathbf{x} + \mathbf{M}^{\mathrm{T}}[\widehat{\mathbf{M}\mathbf{e}}]^{-1}\mathbf{f} \tag{1.100}$$

From Eqs. 1.100 and 1.22 it follows that $\mathbf{f}^{*} = \mathbf{M}^{\mathrm{T}}[\widehat{\mathbf{M}\mathbf{e}}]^{-1}\mathbf{f}$. Hence, under the FPSA, the joint production system described by Eqs. 1.20 and 1.21 is converted into the single production system described by the following equations

$$\mathbf{M}^{\mathrm{T}}\mathbf{e} = \mathbf{A}(\mathbf{U},\mathbf{M})\mathbf{M}^{\mathrm{T}}\mathbf{e} + \mathbf{M}^{\mathrm{T}}[\widehat{\mathbf{Me}}]^{-1}\mathbf{f} \tag{1.101}$$

and

$$\mathbf{e}^{\mathrm{T}}\mathbf{M} = \mathbf{e}^{\mathrm{T}}\mathbf{A}(\mathbf{U},\mathbf{M})[\widehat{\mathbf{e}^{\mathrm{T}}\mathbf{M}}] + \mathbf{v}^{\mathrm{T}} \tag{1.102}$$

Comparing Eqs. 1.42 and 1.43 with Eqs. 1.101 and 1.102, i.e. the single production system derived under the ITA with the single production system derived under the FPSA, it follows that

$$\mathbf{A}_{\mathrm{I}}(\mathbf{U},\mathbf{M}) = \mathbf{S}\mathbf{A}_{\mathrm{C}}(\mathbf{U},\mathbf{M})\mathbf{S}^{-1}$$

$$\mathbf{f}_{\mathrm{I}}^{*} = \mathbf{S}\mathbf{f}_{\mathrm{C}}^{*}$$

$$\boldsymbol{\pi}_{\mathrm{I}}^{*\mathrm{T}} = \boldsymbol{\pi}_{\mathrm{C}}^{*\mathrm{T}}\mathbf{S}^{-1}$$

and

$$\mathbf{x}_{\mathrm{I}} = \mathbf{S}\mathbf{x}_{\mathrm{C}}$$

where $\mathbf{S} \equiv \mathbf{M}^{\mathrm{T}}[\widehat{\mathbf{Me}}]^{-1}$, while $\mathbf{A}_{\mathrm{I}}(\mathbf{U},\mathbf{M})$ ($\mathbf{A}_{\mathrm{C}}(\mathbf{U},\mathbf{M})$), $\mathbf{f}_{\mathbf{I}}^{*}$ ($\mathbf{f}_{\mathbf{C}}^{*}$), $\boldsymbol{\pi}_{\mathrm{I}}^{*\mathrm{T}} \equiv [\mathbf{v}_{\mathrm{I}}^{*\mathrm{T}}\hat{\mathbf{x}}_{\mathrm{I}}^{-1}]$ ($\boldsymbol{\pi}_{\mathrm{C}}^{*\mathrm{T}} \equiv [\mathbf{v}_{\mathrm{C}}^{*\mathrm{T}}\hat{\mathbf{x}}_{\mathrm{C}}^{-1}]$) and $\mathbf{x}_{\mathrm{I}}$ ($\mathbf{x}_{\mathrm{C}}$) is the technical coefficients matrix, the transformed vector of final demand, the transformed vector of value added coefficients and the transformed vector of value added, respectively, derived under the FPSA (ITA). These relationships imply that the FPSA (ITA) can be derived from ITA (FISA) via a similarity transformation.

1.4.3 Evaluation of the Conversion Methods

The next issue that comes up is which of the conversion methods is the most suitable for the problem at hand. Since there were not any objective criteria to test the consistency of the various methods, Jansen and ten Raa (1990) developed four desirable properties or, alternative, axioms that the various methods converting SUTs into product-by-product SIOTs should fulfill. These properties are:

(i). The "material balance property" or, in formal terms,

$$\mathbf{A}(\mathbf{U},\mathbf{M})\mathbf{Me} = \mathbf{Ue}$$

This property implies that the requirements needed to produce the output should be equal to the actual inputs of the economy.

(ii). The "financial balance property" or, in formal terms,

$$\mathbf{e}^{\mathrm{T}}\mathbf{A}(\mathbf{U},\mathbf{M})\mathbf{M} = \mathbf{e}^{\mathrm{T}}\mathbf{U}$$

This property implies that the input cost of the output should be equal to the cost of the actual inputs.

(iii). The "price invariance property" or, in formal terms,

$$\mathbf{A}(\hat{\mathbf{p}}_{\mathrm{b}}\mathbf{U},\hat{\mathbf{p}}_{\mathrm{b}}\mathbf{M}) = \hat{\mathbf{p}}_{\mathrm{b}}\mathbf{A}(\mathbf{U},\mathbf{M})\hat{\mathbf{p}}_{\mathrm{b}}^{-1}, \ \forall \mathbf{p}_{\mathrm{b}} > \mathbf{o}$$

where $\mathbf{p}_{\mathrm{b}}$ is the price vector relative to the base-year prices. This property implies that whatever the base-year price is, the corresponding technical coefficients matrix should be similar to the matrix $\mathbf{A}(\mathbf{U},\mathbf{M})$.

(iv). The “scale invariance property” or, in formal terms,

$$\mathbf{A}(\mathbf{U}\hat{\mathbf{s}},\mathbf{M}\hat{\mathbf{s}})=\mathbf{A}(\mathbf{U},\mathbf{M}),\ \forall \mathbf{s}>\mathbf{0}$$

This property guarantees that the technical coefficients matrix does not depend on the activity levels of the economy.

Jansen and ten Raa (1990) proved that: (i) the PTA fulfils all the desirable properties; (ii) the ITA fulfils only the material balance property”;[18] (iii) the by-product method and the mixed technology model, constructed by ten Raa et al. (1984), fulfil the price and scale invariance properties; (iv) the transfer method does not fulfil any of the properties; (v) the ESA method fulfils the material balance and price invariance properties; and (vi) the lump-sum method fulfils only the scale invariance property.

Following Jansen and ten Raa (1990), Rueda-Cantuche and ten Raa (2009) developed four desirable properties that the conversion methods that derive industry-by-industry SIOTs should fulfill, which are conjugated to those developed by Jansen and ten Raa (1990) for the conversion methods that derive product-by-product SIOTs. These properties are:

(i). The material balance property:

$$\mathbf{M}[\widehat{\mathbf{e}^{\mathrm{T}}\mathbf{M}}]^{-1}\mathbf{A}(\mathbf{U},\mathbf{M})[\widehat{\mathbf{e}^{\mathrm{T}}\mathbf{M}}]\mathbf{e}=\mathbf{U}\mathbf{e}$$

(ii). The financial balance property:

$$\mathbf{e}^{\mathrm{T}}\mathbf{M}[\widehat{\mathbf{e}^{\mathrm{T}}\mathbf{M}}]^{-1}\mathbf{A}(\mathbf{U},\mathbf{M})[\widehat{\mathbf{e}^{\mathrm{T}}\mathbf{M}}]=\mathbf{e}^{\mathrm{T}}\mathbf{U}$$

(iii). The price invariance property:

$$\mathbf{A}(\hat{\mathbf{p}}_{\mathrm{b}}\mathbf{U},\hat{\mathbf{p}}_{\mathrm{b}}\mathbf{M})=\hat{\mathbf{p}}_{\mathrm{b}}\mathbf{M}[\widehat{\mathbf{e}^{\mathrm{T}}\mathbf{M}}]^{-1}\mathbf{A}(\mathbf{U},\mathbf{M})[\widehat{\mathbf{e}^{\mathrm{T}}\mathbf{M}}][\widehat{\mathbf{p}_{\mathrm{b}}\mathbf{M}}]^{-1},\ \forall \mathbf{p}_{\mathrm{b}}>\mathbf{0}$$

(iv). The scale invariance property:

$$\mathbf{A}(\mathbf{U}\hat{\mathbf{s}},\mathbf{M}\hat{\mathbf{s}})=\mathbf{A}(\mathbf{U},\mathbf{M}),\ \forall \mathbf{s}>\mathbf{0}$$

Rueda-Cantuche and ten Raa (2009) proved that the FISA fulfils all the desirable properties, while the FPSA fulfils only the financial balance property.

Thus, it is concluded that only the PTA and the FISA fulfil all the desirable properties. However, both the PTA and the FISA can be criticized because (i) they cannot be applied to the case of rectangular SUTs;[19] and (ii) the technical coefficients matrix that is derived from these methods is possible to contain negative elements. In order to overcome the problem of negative coefficients, there have been proposed various procedures for removing the negative coefficients that may appear in the technical coefficients matrix.[20] A well-known

[18] ten Raa et al. (1984) had shown that the technical coefficients matrix derived under the ITA depends on the base year prices. In other words, they had proved that the ITA does not fulfil the price invariance property.

[19] That is because those methods require the inversion of the make matrix.

[20] For a detailed review of the available methods to remove negative coefficients, see ten Raa and Rueda-Cantuche (2005, pp. 4–13).

method is that proposed by Almon (1970; 2000), which consists of an iterative procedure of changes in the technical coefficients matrix that converges to a (semi-)positive matrix, but convergence is guaranteed only if more than half of the production of each commodity is in its primary industry. However, Almon's method has been criticized for being without economic justification (see ten Raa et al. 1984, p. 93; ten Raa and Rueda-Cantuche 2013). Alternatively, mixed technology models are often used in order to overcome the problem of negative coefficients (see, e.g. Armstrong 1975). Nevertheless, mixed technology models cannot guarantee the derivation of a technical coefficients matrix with non-negative coefficients. Moreover, de Mesnard (2011) has pointed out that even in the case where the technical coefficients matrix derived from PTA or FISA is non-negative, both methods should be rejected as economically irrelevant.

On the basis of the previous analysis it can be concluded that none of the conversion methods can guarantee (i) consistency with the requirements of input–output analysis; and (ii) economically acceptable results. We may find a way out of that problem by accepting that we live in a world where joint production economic activities are common and by making use of general joint production models inspired by von Neumann (1937; 1945) and Sraffa (1960). In the next section we present the essential ideas of the von Neumann–Sraffa-based approach to the case of joint production as a preferable approach to treat SUTs.[21]

1.5 The von Neumann–Sraffa-Based Approach

A square linear system of joint production *à la* von Neumann–Sraffa is defined by the pair $\{\mathbf{B}, \mathbf{A}\}$, where $\mathbf{B}$ is the output matrix and $\mathbf{A}$ is the input matrix (both $\mathbf{B}$ and $\mathbf{A}$ are expressed in physical terms). Also, let $\mathbf{d}$ be the vector of final demand (in physical terms), $\boldsymbol{\pi}$ be the vector of value added coefficients, $\mathbf{y}$ be the vector of activity levels, and $\mathbf{p}$ be the vector of market prices. Then we can write

$$\mathbf{By} = \mathbf{Ay} + \mathbf{d} \tag{1.103}$$

and

$$\mathbf{p}^{\mathrm{T}}\mathbf{B} = \mathbf{p}^{\mathrm{T}}\mathbf{A} + \boldsymbol{\pi}^{\mathrm{T}} \tag{1.104}$$

The above system is said to be strictly *viable* if it can produce a physical net surplus of any commodity or, in formal terms,

$$\exists \mathbf{y} \geq \mathbf{0}, \ [\mathbf{B} - \mathbf{A}]\mathbf{y} > \mathbf{0}$$

[21] For a detailed exposition of the von Neumann–Sraffa-based analysis and the connection between the works of von Neumann and Sraffa, see Kurz and Salvadori (1995, Chap. 8 and pp. 403–426; 2001).

A system $\{\mathbf{B},\mathbf{A}\}$ is said to be strictly *profitable* if there exists a price vector $\mathbf{p}$ for which all industries are profitable, or, in formal terms,[22]

$$\exists \mathbf{p} \geq \mathbf{0}\ ,\ \mathbf{p}^{\mathrm{T}}[\mathbf{B}-\mathbf{A}] > \mathbf{0}^{\mathrm{T}}$$

A commodity i is said to be *separately producible* if it is possible to produce a net output consisting of a unit of that commodity alone with a nonnegative vector of activity levels or, in formal terms,

$$\exists \mathbf{y} \geq \mathbf{0},\ [\mathbf{B}-\mathbf{A}]\mathbf{y} = \mathbf{e}_i$$

where $\mathbf{e}_i$ is a vector whose ith element is equal to 1 and all other elements are equal to zero. A system $\{\mathbf{B},\mathbf{A}\}$ is said to be *all-productive* if all products are separately producible or, in formal terms,

$$\forall \mathbf{d} \geq \mathbf{0}\ ,\ \exists \mathbf{y} \geq \mathbf{0},\ [\mathbf{B}-\mathbf{A}]\mathbf{y} = \mathbf{d}$$

Thus, if $\{\mathbf{B},\mathbf{A}\}$ is all-productive then $[\mathbf{B}-\mathbf{A}]^{-1} \geq \mathbf{0}$ (and *vice versa*). A process, within a system $\{\mathbf{B},\mathbf{A}\}$, is called *indispensable* if it has to be activated whatever net output is to be produced. An all-productive system whose processes are all indispensable is called *all-engaging*. Formally, the system $\{\mathbf{B},\mathbf{A}\}$ is all-engaging iff the following two properties hold

$$\exists \mathbf{y} \geq \mathbf{0},\ [\mathbf{B}-\mathbf{A}]\mathbf{y} \geq \mathbf{0}$$

$$\{\exists \mathbf{y} \geq \mathbf{0}, [\mathbf{B}-\mathbf{A}]\mathbf{y} \geq \mathbf{0}\} \Rightarrow \mathbf{y} > \mathbf{0}$$

Thus, if $\{\mathbf{B},\mathbf{A}\}$ is all-engaging then $[\mathbf{B}-\mathbf{A}]^{-1} > \mathbf{0}$ (and *vice versa*). As is well known, the concepts of "all-productive" ("all-engaging") systems correspond with systems that retain all the essential properties of reducible (irreducible) single-product systems.[23]

We now return to the actual economic system described by the make and use matrices, i.e. the pair $\{\mathbf{M},\mathbf{U}\}$. The make and use matrices can be rewritten as

$$\mathbf{M} = \hat{\mathbf{p}}\mathbf{B}\hat{\mathbf{y}} \tag{1.105}$$

and

$$\mathbf{U} = \hat{\mathbf{p}}\mathbf{A}\hat{\mathbf{y}} \tag{1.106}$$

Analogously, the vectors of final demand and value added can be rewritten as

$$\mathbf{f} = \hat{\mathbf{p}}\mathbf{d} \tag{1.107}$$

and

$$\mathbf{v}^{\mathrm{T}} = \boldsymbol{\pi}^{\mathrm{T}}\hat{\mathbf{y}} \tag{1.108}$$

[22] In the case of joint production, the conditions of viability and profitability are not equivalent (see Bidard 1986, pp. 55–56). It need hardly be said that, in general, none of the usual laws of single production systems hold true in the case of joint production (see Bidard 1997; Steedman 1982).
[23] See Bidard (1996); Kurz and Salvadori (1995, pp. 238–240); Schefold (1978).

Now, we assume that the physical unit of measurement of each product is that unit which is worth of a monetary unit, i.e. it holds

$$\mathbf{p} = \mathbf{e} \tag{1.109}$$

Substituting Eqs. 1.105, 1.106, 1.107 and 1.109 into Eq. 1.20 we obtain

$$\mathbf{B}\hat{\mathbf{y}}\mathbf{e} = \mathbf{A}\hat{\mathbf{y}}\mathbf{e} + \mathbf{d}$$

or

$$\mathbf{B}\mathbf{y} = \mathbf{A}\mathbf{y} + \mathbf{d} \tag{1.110}$$

Finally, substituting Eqs. 1.105, 1.106, 1.108 and 1.109 into Eq. 1.21 we obtain

$$\mathbf{e}^{\mathrm{T}}\mathbf{B}\hat{\mathbf{y}} = \mathbf{e}^{\mathrm{T}}\mathbf{A}\hat{\mathbf{y}} + \boldsymbol{\pi}^{\mathrm{T}}\hat{\mathbf{y}}$$

or

$$\mathbf{e}^{\mathrm{T}}\mathbf{B} = \mathbf{e}^{\mathrm{T}}\mathbf{A} + \boldsymbol{\pi}^{\mathrm{T}} \tag{1.111}$$

Thus, it follows that Eq. 1.110 is equivalent to Eq. 1.103 and, by taking into account Eq. 1.109, it follows that Eq. 1.111 is equivalent to Eq. 1.104. Consequently, the system described by the make and use matrices can be considered as the empirical counterpart of a joint production system *à la* von Neumann–Sraffa. Namely, the make matrix, $\mathbf{M}$, can be considered as the counterpart of the output matrix, $\mathbf{B}$, and the use matrix, $\mathbf{U}$, can be considered as the counterpart of the matrix $\mathbf{A}$ (also see Bidard and Erreygers 1998, pp. 434–436). Therefore, an actual economy will be said to be all-productive (all-engaging) when it holds $[\mathbf{M}-\mathbf{U}]^{-1} \geq \mathbf{0}$ ($[\mathbf{M}-\mathbf{U}]^{-1} > \mathbf{0}$).[24]

The conversion methods try to transform the system described by the pair of matrices $\{\mathbf{M},\mathbf{U}\}$ into the single production system described by the pair of matrices $\{\mathbf{I},\mathbf{A}(\mathbf{U},\mathbf{M})\}$. This means that all conversion methods assume, implicitly or otherwise, that the there is a single production system "hidden", i.e. not directly observable, in the SUTs. However, this is a groundless assumption. The awareness of joint product processes has been already familiar to classical economists, such as Adam Smith (1776, Book 1, Chap. 11).[25] For instance, Jevons ([1871] 1888, Chap. 5) points out that the cases of joint production form the general rule, to which it is difficult to find important exceptions, while similar points have been stressed more recently (see Baumgärtner et al. 2006; Faber et al. 1998; Kurz 2006; Steedman 1984). Contrary to the approach imposed by the conversion (namely, transmutation) methods, the von Neumann–Sraffa-based analysis of joint production constitutes a straightforward approach, i.e. it does not rule out joint production, which is not based on any of the restrictive (and debatable) assumptions of the conversion methods.

[24] The relevant empirical investigation so far has shown that none of the actual systems is all-productive (see Chaps. 5, 7 and 9).

[25] For a review of the contributions of classical and early neoclassical economists to the analysis of joint production, see Kurz (1986).

To sum up, given that (i) the pair $\{\mathbf{M},\mathbf{U}\}$ can be considered as the empirical counterpart of the pair $\{\mathbf{B},\mathbf{A}\}$; and (ii) joint production constitutes the empirically relevant case, it would seem reasonable that a straightforward treatment of SUTs, based on the von Neumann–Sraffa-based analysis, to be preferred instead of trying to derive single production tables (i.e. SIOTs).

1.6 Concluding Remarks

This chapter mapped the structure of the empirical input–output representations of actual economies, i.e. the Supply and Use Tables (SUTs) and the Symmetric Input–Output Tables (SIOTs). It has been shown that the main difference between these two types of input–output tables is that the SUTs, which constitute the core of the modern systems of national accounting, allow for joint production activities, whereas the SIOTs rule out joint production and are constructed on the basis of specific assumptions on the relationships between inputs and outputs recorded in the SUTs. The review of the alternative methods used to convert SUTs into SIOTs revealed that, despite the differences amongst those methods, they all rest on the tacit assumption that there is a single production system hidden in the SUTs characterizing the real world. It has been argued that this is a groundless assumption, and that a consistent approach is the straightforward treatment of SUTs on the basis of general joint production models inspired by the von Neumann and Sraffa contributions.

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