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**ECONOMETRICS**
**Problem Set 3**
Due by: Tuesday May 16, 2023, at 6pm.

Instructions: You may work alone or in teams of 2, but everyone must submit their own homework. The HW is due in class Tuesday May 16.

**(1)** (The Curved Roof Distribution). The pdf of the *Curved-Roof* distribution is given by

$$f_{X,Y}(x,y) = \frac{3}{11}(x^2 + y), \quad \text{for } x \in [0,2], \text{and } y \in [0,1].$$

(i) Plot $f_{X,Y}(x,y)$.
(ii) Derive the marginal pdf of $X$, $f_X(x)$, the marginal pdf of $Y$, $f_Y(y)$, and plot them.
(iii) Using the marginal pdf's, find $E(X)$, $E(Y)$, $V(X)$, and $V(Y)$.
(iv) Derive the conditional pdf of $Y|X$, $f_{Y|X}(y|x)$, the conditional pdf of $X|Y$, $f_{X|Y}(x|y)$, and plot them.
(v) Derive the CEF of $Y|X$, $E(Y|X)$, and the CEF of $X|Y$, $E(X|Y)$.
(vi) Find the BLP of $Y|X$, $L(Y|X)$, and the BLP of $X|Y$, $L(X|Y)$.
(vii) Plot $E(Y|X)$ along with $L(Y|X)$, indicating with a horizontal line in your graph the position of $E(Y)$. Repeat for $E(X|Y), L(X|Y)$, and $E(X)$.

**(2)** Consider two Poisson random variables, $Y \sim Poisson(\lambda_1)$ and $U \sim Poisson(\lambda_2)$, i.e.

$$f_Y(y) = e^{-\lambda_1}\frac{\lambda_1^y}{y!}, \quad \text{for } y = 0, 1, 2, ...$$

and similarly for $U$, and let $X = Y + U$.

(i) Show that X $\sim Poisson(\lambda_1 + \lambda_2)$.

(ii) Show that the joint pmf of $X$ and $Y$, is given by

$$f_{X,Y}(x,y) = \Pr(X = x, Y = y) = e^{-(\lambda_1+\lambda_2)}\frac{\lambda_2^{x-y}\lambda_1^y}{(x-y)!y!},$$

for $x = 0, 1, ...$ and $y = 0, 1, ..., x$. Explain the restriction in the support.

(iii) Show that the conditional pmf of $Y$ given $X = x$ is

$$f_{Y|X}(y|x) = Binomial\Big(x, \frac{\lambda_1}{\lambda_1 + \lambda_2}\Big),$$

where $Binomial(n, p)$ denotes the pmf of a binomial over $n$ trials and probability of success $p$. Find $E(Y|X)$ and $V(Y|X)$.

(iv) Are $X$ and $Y$ independent? Prove your claim.

**(3)** (A Mixture Model). A population is composed of 50% of men, and 50% of women. We know that heights (in cm's) are normally distributed in both sub-populations of men and women, with men's heights distributed as $N(177, (6.5)^2)$, and women's heights distributed as $N(165, (5.5)^2)$. Let $X$ be a categorical variable that equals 1 if a person is man and 0 if it is a woman, and let $Y$ denote height.

(i) Find the pdf of heights in the population, $f_Y(y)$, and plot it.

(ii) If we randomly choose someone and find that the person's height is 170cm, what is the probability that this person is a man, i.e., find $\Pr(X = 1|Y = 170)$?