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ECONOMETRICS
Problem Set 2

Due by: Sat. April 4, 2023, at 6pm.

Instructions: You may work alone or in teams of 2, but everyone must submit their own homework. The HW should be scanned in a pdf and emailed to my account `gkordas@panteion.gr` by 6pm of the due date.

(1) Show that if $\theta \sim U(-\pi/2, \pi/2)$, then $X = \tan\theta \sim$ Cauchy, i.e., show that

$$f(x) = \frac{1}{\pi(1+x^2)}.$$

[This result provides an easy way to generate realizations from a Cauchy random variable in a computer.]

(2) Let $X \sim \gamma(p, \lambda)$ with density

$$f(x) = \frac{\lambda^p}{\Gamma(p)} x^{p-1} e^{-\lambda x}, \quad x > 0, p > 0, \lambda > 0.$$

(i) Show that it's moment generating function is given by

$$\psi(t) = E(e^{tX}) = \Big(\frac{\lambda}{\lambda - t}\Big)^p, \quad t < \lambda.$$

Hint: Use the Gamma integral

$$\int_0^\infty e^{-ax} x^{n-1} dx = \frac{\Gamma(n)}{a^n}.$$

(ii) Show that

$$E(X^r) = \frac{\Gamma(r+p)}{\lambda^r \Gamma(p)}.$$

(iii) Find $E(X)$ and $V(X)$.

(3) Let $X \sim \beta(p, q)$ with density

$$f(x) = \frac{1}{B(p,q)} x^{p-1}(1-x)^{p-1}, \quad x > 0, p > 0, q > 0,$$

where

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

(i) Show that

$$E(X^r) = \frac{p\cdots(p+r-1)}{(p+q)\cdots(p+q+r-1)}.$$

Hint: Use the Beta integral

$$B(p,q) = B(q,p) = \int_0^1 x^{p-1}(1-x)^{q-1}dx.$$

(ii) Show that

$$V(X) = \frac{pq}{(p+q)^2(p+q+1)}.$$

(4) Let X and Y be independent Exponential(1) random variables each having p.d.f.

$$f(x) = e^{-x}, \quad x > 0.$$

Show that $Z = Y - X$ is distributed as a double exponential with density

$$f(z) = \frac{1}{2}e^{-|z|}, \quad -\infty < z < \infty.$$

You may take as given that the moment generating function of the double exponential is

$$\psi_Z(t) = \frac{1}{1-t^2}, \quad |t| < 1.$$

(5) Do Exercises **2.11**, **2.12**, **3.1**, and **3.7** in the *Goldberger* textbook.